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$\Rightarrow (3xy+y^2)+(x^2+xy)y'(x)=0$. Solution is the curve $x^3y+\frac{x^2y^2}{2}+C=0$.

We can check that $\mu(x,y)=\frac{1}{xy(2x+y)}$ is also a integrating factor. ////

Summary: In this Chapter 2, we deal with three type of O.D.E.

I. $y'(t)+p(t)y(t)=q(t)$ (linear O.D.E.) Let $P(t)$ be an anti-derivative of $p(t)$ $\Rightarrow \frac{\partial}{\partial t}\left(e^{P(t)}y(t)\right)=e^{P(t)}q(t)$.

II. $y'(t)=g(y(t))h(t)$ (separable O.D.E.) Let G be an anti-derivative of $\frac{1}{g}$ $\Rightarrow \frac{\partial}{\partial t}(G(y(t)))=h(t)$.

III. $M(x,y)+N(x,y)\frac{dy}{dx}=0$. Find integrating factor.

Important concept: Integral curve. Let $F=(F_1,\cdots,F_n)$ be a vector field. A curve $x(t)=(x_1(t),\cdots,x_n(t))$ is called an integral curve of F if $x_1'(t)=F_1(x_1(t),\cdots,x_n(t)),\cdots,x_n'(t)=F_n(x_1(t),\cdots,x_n(t))$. We need to know how to find integral curve for $n=2$. For $n=2$, $\begin{cases} x'(t)=F(x(t),y(t)) \\ y'(t)=G(x(t),y(t)) \end{cases}$ [*]

$\Rightarrow \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{G(x,y)}{F(x,y)}$. [*] is reduced to $\frac{dy}{dx}F(x,y)+G(x,y)=0$. ////

§3. Second Order linear equations. We first introduce some definition

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Def 3.1: A second order ordinary differential equation has the form $f(t, y, y'(t), y''(t)) = 0$ [※] for some (smooth) function f. The O.D.E [※] is called linear if f takes the form $f(t, y, y'(t), y''(t)) = P(t)y''(t) + Q(t)y'(t) + R(t)y(t) - G(t)$ [※※], where P is a function which never vanishes. The O.D.E [※] is called nonlinear if it is not linear. The functions P, Q, R are called the coefficients of th O.D.E and G is called the forcing of the O.D.E. The initial condition for [※] is $(y(t_0), y'(t_0), y''(t_0))$. ☒

§3.1: Homogeneous equations with constant coefficients

Def 3.2: The O.D.E [※※] is called homogeneous if $G \equiv 0$, otherwise it is called homogeneous. When $G \not\equiv 0$, the term $G(t)$ in [※※ is called the non-homogeneous term. ☒

In this section, we consider homogeneous second order linear O.D.E with constant coefficients: $Py''(t) + Qy'(t) + Ry(t) = 0$, where P, Q, R are independent of t and $P \neq 0$.

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Since $P \equiv 0$, the O.D.E reduces to $y'' + by' + cy = 0$ [*I]. Let λ be the solution to the equation $\lambda^2 + b\lambda + c = 0$.

Case I. Suppose that there are two distinct real roots λ_1 and λ_2.

$\Rightarrow \lambda^2 + b\lambda + c = (\lambda - \lambda_1)(\lambda - \lambda_2) \Rightarrow \left(\frac{\partial}{\partial t} - \lambda_1\right)\left(\frac{\partial y}{\partial t} - \lambda_2 y\right) = \frac{\partial^2 y}{\partial t^2} - \lambda_2 \frac{\partial y}{\partial t} - \lambda_1 \frac{\partial y}{\partial t} + \lambda_1 \lambda_2 y = y'' + by' + cy = 0$. $\Rightarrow y'(t) - \lambda_2 y = 0$ or $z'(t) - \lambda_1 z(t) = 0$ where $z(t) = y'(t) - \lambda_2 y$. If $y'(t) - \lambda_2 y(t) = 0 \Rightarrow y(t) = C_1 e^{\lambda_2 t}$. If $z'(t) - \lambda_1 z(t) = 0$, then $z(t) = C_1 e^{\lambda_1 t} = y'(t) - \lambda_2 y$. Solve $y'(t) - \lambda_2 y = C_1 e^{\lambda_1 t} \Rightarrow e^{-\lambda_2 t} y(t) - e^{-\lambda_2 t} \lambda_2 y = C_1 e^{(\lambda_1 - \lambda_2)t} = \frac{\partial}{\partial t}\left(e^{-\lambda_2 t} y(t)\right) = C_1 e^{(\lambda_1 - \lambda_2)t} \Rightarrow e^{-\lambda_2 t} y(t) = C_1 \int_0^t e^{(\lambda_1 - \lambda_2)\tau} d\tau + \hat{C}_2 = \frac{C_1 e^{(\lambda_1 - \lambda_2)t}}{\lambda_1 - \lambda_2} + C_2 \Rightarrow y(t) = \frac{C_1 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} + C_2 e^{\lambda_2 t}$. In other words, a solution of [*I] is a linear combination of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ if λ_1 and λ_2 are distinct real roots of $\lambda^2 + b\lambda + c = 0$.

Case II. Suppose that there is a real double root of $\lambda \Rightarrow (e^{-\lambda t} y)' = -\lambda e^{-\lambda t} y + e^{-\lambda t} y'$. $\Rightarrow (e^{-\lambda t} y)'' = \lambda^2 e^{-\lambda t} y - \lambda e^{-\lambda t} y' - \lambda e^{-\lambda t} y' + e^{-\lambda t} y'' = e^{-\lambda t}(y'' - 2\lambda y' + \lambda^2 y) = e^{-\lambda t}(y'' + by' + cy) = 0 \Rightarrow (e^{-\lambda t} y)' = C_2 \Rightarrow e^{-\lambda t} y = C_2 t + C_1$ $\Downarrow$ $y = C_2 t e^{\lambda t} + C_1 e^{\lambda t}$. In other words, a solution of [*I] is a linear combination of $te^{\lambda t}$ and $e^{\lambda t}$. 「Q: If there are complex roots for $\lambda^2 + b\lambda + c = 0$?」

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Def 3.3: The characteristic equation for the O.D.E. [※I] is $\lambda^2+b\lambda+$

Consider $y''+by'+cy=0$. Let $z=y'$. Then $\begin{pmatrix}y'\\z'\end{pmatrix}=\begin{pmatrix}z\\-by'-cy\end{pmatrix}=\begin{pmatrix}0&1\\-c&-b\end{pmatrix}\begin{pmatrix}y\\z\end{pmatrix}$

Let $A=\begin{pmatrix}0&1\\-c&-b\end{pmatrix}$, $Y=\begin{pmatrix}y\\z\end{pmatrix}\Rightarrow Y'=AY$. The characteristic equation ⋯ the matrix A is given by $\det(\lambda I-A)=\det\begin{pmatrix}\lambda&-1\\c&\lambda+b\end{pmatrix}=\lambda^2+b\lambda+c=0$

Suppose that $A=P\Lambda P^{-1}$ for some diagonal matrix Λ. $\Lambda=\begin{pmatrix}\lambda_1&0\\0&\lambda_2\end{pmatrix}\Rightarrow$

$Y'=P\Lambda P^{-1}Y\Rightarrow P^{-1}Y'=\Lambda P^{-1}Y$. Let $P^{-1}Y=U=\begin{pmatrix}u_1\\u_2\end{pmatrix}\Rightarrow U'=\Lambda U\Rightarrow\begin{pmatrix}u_1'\\u_2'\end{pmatrix}=$

$\begin{pmatrix}\lambda_1&0\\0&\lambda_2\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}\Rightarrow u_1=c_1e^{\lambda_1 t},\ u_2=c_2e^{\lambda_2 t}\Rightarrow Y=PU=\begin{pmatrix}e&g\\f&h\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}y\\y'\end{pmatrix}\Rightarrow\begin{pmatrix}y\\y'\end{pmatrix}$

$=\begin{pmatrix}e&g\\f&h\end{pmatrix}\begin{pmatrix}c_1e^{\lambda_1 t}\\c_2e^{\lambda_2 t}\end{pmatrix}\Rightarrow y$ is a linear combination of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$.

Consider $A=\begin{pmatrix}0&1\\-c&-b\end{pmatrix}$. If $\lambda^2+b\lambda+c=0$ has two distinct real roots λ_1,λ_2, we can find $\Lambda=\begin{pmatrix}\lambda_1&0\\0&\lambda_2\end{pmatrix}$ such that $A=P\Lambda P^{-1}$. If $\lambda^2+b\lambda+c=0$ has a real double root λ, then $A=P\underbrace{\begin{pmatrix}\lambda&1\\0&\lambda\end{pmatrix}}_{\text{Jordan form}}P^{-1}\Rightarrow U'=\begin{pmatrix}\lambda&1\\0&\lambda\end{pmatrix}U\Rightarrow$

$\begin{pmatrix}u_1'\\u_2'\end{pmatrix}=\begin{pmatrix}\lambda&1\\0&\lambda\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}\Rightarrow\begin{matrix}u_1'=\lambda u_1+u_2\\u_2'=\lambda u_2\end{matrix}\Rightarrow u_2=e^{\lambda t}c\Rightarrow u_1'=\lambda u_1+e^{\lambda t}c\Rightarrow$

$u_1=cte^{\lambda t}\Rightarrow y$ is a linear combination of $e^{\lambda t}$ and $te^{\lambda t}$.

→插入[完]

§3.2: Solution to Second order linear homogeneous equations, the Wronsk

Consider the O.D.E $L[y] := y'' + p(t)y' + q(t)y = 0$ with initial condition $y(t_0) = y_0$ and $y'(t_0) = y_1$.

Thm 3.4: Consider the initial value problem $y'' + p(t)y' + q(t)y = g(t)$, $y(t_0) = y_0$, $y'(t_0) = y_1$, where p, q, g are continuous on an open interval I that contains the point t_0. Then there is exactly one solution $y = \varphi(t)$ of this problem and the solution exists throughout the interval I.

In the following, we assume that p, q are continuous in the interval of interests.

Thm 3.5 (Principle of Superposition): If $y = \varphi_1$ and $y = \varphi_2$ are two solutions of the O.D.E $L[y] := y'' + py' + qy = 0$, then the linear combination $c_1\varphi_1 + c_2\varphi_2$ is also a solution of $L[y]$ for any constants c_1 and c_2. In other words, the collection of the solutions to $L[y]$ is a vector space.

Question: Given two solutions $y = \varphi_1$ and $y = \varphi_2$ of $L[y] = y'' + py' + qy = 0$ with initial condition $y(t_0) = y_0$, $y'(t_0) = y_1$, can be written as a linear

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combination of φ_1 and φ_2 (for whatever given initial data)? If this is true, then $\underbrace{\{\text{the vector space consisting of solutions to } L[y]\}}_{\text{called the solution space}}$ is two dimensional if φ_1 and φ_2 are linear independent and $\{\varphi_1, \varphi_2\}$ is a basis of the solution space of $L[y]$.

[卍] If $\lambda^2+b\lambda+c=0$ has two complex roots $\lambda \pm i\mu$, $\lambda, \mu \in \mathbb{R}$, $\mu \neq 0$,

Then $A = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix} = P \underbrace{\begin{pmatrix} \lambda+i\mu & 0 \\ 0 & \lambda-i\mu \end{pmatrix}}_{\Lambda} P^{-1}$, P and P^{-1} are complex matrices

$\Rightarrow Y' = P\Lambda P^{-1}Y \Rightarrow P^{-1}Y' = \Lambda P^{-1}Y$. Let $P^{-1}Y = U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Rightarrow \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} \lambda+i\mu & 0 \\ 0 & \lambda-i\mu \end{pmatrix}$

$\Rightarrow u_1' = (\lambda+i\mu)u_1$, $u_2' = (\lambda-i\mu)u_2 \Rightarrow u_1 = c_1 e^{(\lambda+i\mu)t} = c_1 e^{\lambda t}(\cos\mu t + i\sin\mu t)$,

$u_2 = c_2 e^{\lambda t}(\cos\mu t - i\sin\mu t) \Rightarrow y$ is a linear combination of $e^{\lambda t}(\cos\mu t + i\sin\mu t)$ and $e^{\lambda t}(\cos\mu t - i\sin\mu t) \Leftrightarrow y$ is a linear combination of $e^{\lambda t}\cos\mu t$ and $e^{\lambda t}\sin\mu t$.

Def 3.6: Let φ_1 and φ_2 be two differentiable functions. The Wronskian or Wronskian determinant of φ_1 and φ_2 at point t_0
(Polish) (Wronski)
is the number $W(\varphi_1, \varphi_2)(t_0) = \det \begin{pmatrix} \varphi_1(t_0) & \varphi_2(t_0) \\ \varphi_1'(t_0) & \varphi_2'(t_0) \end{pmatrix} = \varphi_1(t_0)\varphi_2'(t_0) - \varphi_2(t_0)\varphi_1'(t_0)$

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<u>Thm 3.8</u>: Suppose that $y=\varphi_1$ and $y=\varphi_2$ are two solutions of $L[y] = y''+py'+qy=0$. Then for any arbitrarily given (y_0, y_1), the solution to $L[y]$ with initial condition $y(t_0)=y_0$, $y'(t_0)=y_1$ can be written as a linear combination of φ_1 and φ_2 if and only if $W(\varphi_1,\varphi_2)(t_0)\neq 0$.

<u>pf</u>: "$\Leftarrow$" Since $W(\varphi_1,\varphi_2)(t_0)\neq 0$, there exists $c_1, c_2\in\mathbb{R}$, such that $\begin{pmatrix}\varphi_1(t_0) & \varphi_2(t_0)\\ \varphi_1'(t_0), & \varphi_2'(t_0)\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}=\begin{pmatrix}y_0\\ y_1\end{pmatrix}$. Let $y=c_1\varphi_1+c_2\varphi_2$. Then φ is a solution of $L[y]$ and $y(t_0)=y_0$, $y'(t_0)=y_1$. "$\Rightarrow$" Let $(y_0,y_1)=(0,1)$. Then, there is a $y(t)=c_1\varphi_1+c_2\varphi_2$ such that $y(t_0)=0$, $y'(t_0)=1$. Let $(y_0,y_1)=(1,0)$. Then, there is a $y(t)=c_3\varphi_1+c_4\varphi_d$ such that $y(t_0)=1$, $y'(t_0)=0$ $\Rightarrow$ $\begin{pmatrix}\varphi_1(t_0) & \varphi_2(t_0)\\ \varphi_1'(t_0), & \varphi_2'(t_0)\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}=\begin{pmatrix}0\\ 1\end{pmatrix}$, $\begin{pmatrix}\varphi_1(t_0) & \varphi_2(t_0)\\ \varphi_1'(t_0), & \varphi_2'(t_0)\end{pmatrix}\begin{pmatrix}c_3\\ c_d\end{pmatrix}=\begin{pmatrix}1\\ 0\end{pmatrix}$.

$\Rightarrow$ The vectors $\begin{pmatrix}\varphi_1(t_0)\\ \varphi_1'(t_0)\end{pmatrix}$, $\begin{pmatrix}\varphi_2(t_0)\\ \varphi_2'(t_0)\end{pmatrix}$ are linear independent. $\Rightarrow W(\varphi_1,\varphi_2)(t_0)\neq 0$

<u>Def 3.9</u>: Let φ_1, φ_2 be solutions to $L[y]$. We say that $\{\varphi_1,\varphi_2\}$ is a fundamental set of $L[y]$ if $W(\varphi_1,\varphi_2)(t_0)\neq 0$, for some t_0 in the interval of interest.

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Thm 3.10: Let $L[y] := y'' + p(t)y' + q(t)y = 0$. We can always find a fundamental set of $L[y]$. Hence the solution space of $L[y]$ is two dimensional.

pf: Let $t_0 \in I$. By Thm 3.4, we can find solutions φ_1, φ_2 of $L[y]$ such that $\varphi_1(t_0) = 0$, $\varphi_1'(t_0) = 1$ and $\varphi_2(t_0) = 1$, $\varphi_2'(t_0) = 0$. Then $W(\varphi_1, \varphi_2) \neq 0$. $\{\varphi_1, \varphi_2\}$ is a fundamental set of $L[y]$. By Thm 3.8, every solution to $L[y]$ is a linear combination of φ_1 and φ_2.

Thm 3.11 (Abel): Let φ_1 and φ_2 be solutions of $L[y] := y'' + p(t)y' + q(t)y$ where p, q are continuous in an open interval I. Assume that $W(\varphi_1, \varphi_2) \neq 0$, for some $t_0 \in I$. Then, $W(\varphi_1, \varphi_2)(t) = W(\varphi_1, \varphi_2)(t_0)\, e^{-\int_{t_0}^{t} p(s)\,ds}$, for every $t \in I$.

pf: $\begin{cases} \varphi_1''(t) + p(t)\varphi_1'(t) + q(t)\varphi_1(t) = 0 & \cdots (1) \\ \varphi_2''(t) + p(t)\varphi_2'(t) + q(t)\varphi_2(t) = 0 & \cdots (2) \end{cases}$ $\quad (1) \times \varphi_2(t) - (2) \times \varphi_1(t) \Rightarrow$

$$\varphi_1''(t)\varphi_2(t) - \varphi_2''(t)\varphi_1(t) + p(t)(\varphi_1'(t)\varphi_2(t) - \varphi_2'(t)\varphi_1(t)) = 0.$$

Now, $W(\varphi_1, \varphi_2)(t) = \det\begin{pmatrix} \varphi_1(t) & \varphi_2(t) \\ \varphi_1'(t) & \varphi_2'(t) \end{pmatrix} = \varphi_1\varphi_2' - \varphi_2\varphi_1' \Rightarrow \frac{\partial}{\partial t} W(\varphi_1, \varphi_2)(t)$

$= \varphi_1'\varphi_2' - \varphi_2'\varphi_1' + \varphi_1\varphi_2'' - \varphi_2\varphi_1'' \Rightarrow \dfrac{\partial W(\varphi_1, \varphi_2)(t)}{\partial t} + p(t)\, W(\varphi_1, \varphi_2)(t) = 0.$

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$\Rightarrow \frac{\partial}{\partial t}\left(e^{+\int_{t_0}^{t} p(s)ds}\, W(\varphi_1,\varphi_2)(t)\right) = 0 \Rightarrow e^{\int_{t_0}^{t} p(s)ds}\, W(\varphi_1,\varphi_2)(t) = W(\varphi_1,\varphi_2)(t_0).$

$\Rightarrow W(\varphi_1,\varphi_2)(t) = e^{-\int_{t_0}^{t} p(s)ds}\, W(\varphi_1,\varphi_2)(t_0).$ ▨

§3.3: Complex Roots of the characteristic equation. Consider again the O.D.E $y''+by'+cy=0$ with b, c constants. Assume that $r^2+br+c=0$ has complex roots $\lambda+i\mu$, $\lambda-i\mu$, $\lambda,\mu\in\mathbb{R}$, $\mu\neq 0$. Let $\varphi_1(t)=e^{\lambda t}\cos\mu t$, $\varphi_2(t)=e^{\lambda t}\sin\mu t$. Check: φ_1, φ_2 are solutions of $y''+by'+cy=0$. $\varphi_1'(t)=\lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t$, $\varphi_1''(t)=\lambda^2 e^{\lambda t}\cos\mu t-\lambda\mu e^{\lambda t}\sin\mu t-\mu\lambda e^{\lambda t}\sin\mu t-\mu^2 e^{\lambda t}\cos\mu t$ $= e^{\lambda t}\cos\mu t(\lambda^2-\mu^2)-2\lambda\mu\sin\mu t\, e^{\lambda t}$. $\varphi_2'(t)=\lambda e^{\lambda t}\sin\mu t+\mu e^{\lambda t}\cos\mu t$. $\varphi_2''(t)=$ $\lambda^2 e^{\lambda t}\sin\mu t+\lambda\mu e^{\lambda t}\cos\mu t+\lambda\mu e^{\lambda t}\cos\mu t-\mu^2 e^{\lambda t}\sin\mu t=(\lambda^2-\mu^2)e^{\lambda t}\sin\mu t+2\lambda\mu e^{\lambda t}\cos\mu t$. $\varphi_1''(t)+b\varphi_1'(t)+c\varphi_1(t)=e^{\lambda t}\cos\mu t(\lambda^2-\mu^2)-2\lambda\mu\sin\mu t\, e^{\lambda t}+b.(\lambda e^{\lambda t}\cos\mu t$ $-\mu e^{\lambda t}\sin\mu t)+ce^{\lambda t}\cos\mu t = e^{\lambda t}\cos\mu t(\lambda^2+b\lambda+c-\mu^2)+e^{\lambda t}\sin\mu t(-2\lambda\mu-b\mu)$.

$\lambda+i\mu=\frac{-b+\sqrt{b^2-4c}}{2}$, $\lambda=-\frac{b}{2}$, $\mu=\frac{\sqrt{4c-b^2}}{2}$. $-2\lambda\mu-b\mu=-\mu(2\lambda+b)=0$.

$\lambda^2+b\lambda+c-\mu^2=\frac{b^2}{4}-\frac{b^2}{2}+c-\frac{(4c-b^2)}{4}=0$. Similarly, $\varphi_2''+b\varphi_2'+c\varphi_2=0$.

$W(\varphi_1,\varphi_2)(t)=\det\begin{pmatrix}\varphi_1(t) & \varphi_2(t)\\ \varphi_1'(t) & \varphi_2'(t)\end{pmatrix}=\det\begin{pmatrix}e^{\lambda t}\cos\mu t, & e^{\lambda t}\sin\mu t\\ \lambda e^{\lambda t}\cos\mu t-\mu e^{\lambda t}\sin\mu t, & \lambda e^{\lambda t}\sin\mu t+\mu e^{\lambda t}\cos\mu t\end{pmatrix}$

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$= \mu e^{\lambda t} = e^{2\lambda t}[\lambda \cos\mu t \sin\mu t + \mu \cos^2 \mu t] - e^{2\lambda t}[\lambda \cos\mu t \sin\mu t - \mu \sin^2 \mu t] \neq 0$

Therefore any solution to $y'' + by' + cy = 0$ can be written as a linear combination of φ_1 and φ_2.

Example 3.12: Consider the motion of an object attached to a spri

The dynamics is described by the ODE $mx''(t) = -kx(t) - rx'(t)$, [*]

where m is the mass of the object, k is the Hooke constant o

the spring and r is the friction coefficient.

$x = 0$

Case I. If $r^2 - 4mk > 0$. There are two distinct negative roots $\frac{-r \pm \sqrt{r^2-4}}{2m}$

to the characteristic equation. The solution of [*] can be writ

as $x(t) = C_1 \exp\left(\frac{-r+\sqrt{r^2-4mk}}{2m}t\right) + C_2 \exp\left(\frac{-r-\sqrt{r^2-4mk}}{2m}t\right)$. Then $\lim\limits_{t\to+\infty} x(t$

Case II. If $r^2 - 4mk = 0$. There is one negative double root $\frac{-r}{2m}$ to

characteristic equation. The solution of [*] can be written as

$x(t) = C_1 \exp\left(\frac{-r}{2m}t\right) + C_2 t \exp\left(\frac{-r}{2m}t\right)$. Then $\lim\limits_{t\to\infty} x(t) = 0$.

Case III. If $r^2 - 4mk < 0$. There are two complex roots $\frac{-r \pm i\sqrt{4mk - r^2}}{2m}$ t the characteristic equation. The solution of [*] can be written as $x(t) = C_1 e^{-\frac{rt}{2m}} \cos\left(\frac{\sqrt{4mk-r^2}}{2m} t\right) + C_2 e^{-\frac{rt}{2m}} \sin\left(\frac{\sqrt{4mk-r^2}}{2m} t\right)$.

(a) If $r=0$, the motion of the object is periodic with period $\frac{4m\pi}{\sqrt{4mk-r^2}}$ and is called simple harmonic motion.

(b) If $r \neq 0$, the object oscillates about the point $x=0$ but approaches to zero exponentially.

§3.4: Repeated Roots; Reduction of Order. Consider $y''+by'+cy=0$ with $b^2=4c$. We already know that $\varphi_1 = e^{-\frac{b}{2}t}$, $\varphi_2 = t e^{-\frac{b}{2}t}$ for a fundamental set of $y''+by'+cy=0$. Suppose we are given a solution $\varphi_1(t)$ and we want to find $\varphi_2(t)$. Let $\varphi_2(t) = \underbrace{V(t)}_{\text{the variation of constant}}\varphi_1(t)$. $\Rightarrow \varphi_2'' = V''\varphi_1 + V\varphi_1'' + 2V'\varphi_1'$, $\varphi_2' = V'\varphi_1 + V\varphi_1'$. $\Rightarrow \varphi_2'' + b\varphi_2' + c\varphi_2 = (V''\varphi_1 + V\varphi_1'' + 2V'\varphi_1') + b(V'\varphi_1 + V\varphi_1') + cV\varphi_1 = V\underbrace{(\varphi_1'' + b\varphi_1' + c\varphi_1)}_{=0} + V''\varphi_1 + 2V'\varphi_1' + bV'\varphi_1 = 0$

$\Rightarrow V''\varphi_1 + 2V'\varphi_1' + bV'\varphi_1 = 0 \Rightarrow V''\varphi_1 = 0 \Rightarrow V'' = 0 \Rightarrow V' = C \Rightarrow V = Ct + C_1$.

The idea of the variation of constant can be generalized to homogeneous

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equations with variable coefficients. Suppose that we have found a solution $y=\varphi_1(t)$ to the O.D.E. $y''+p(t)y'+q(t)y=0$, $t\in I$. Assume the another solution is given by $\varphi_2(t)=v(t)\varphi_1(t)$. Then, $(v''\varphi_1+2v'\varphi_1'+v\varphi_1'')$ $+p(v'\varphi_1+v\varphi_1')+qv\varphi_1=0 \Rightarrow (\varphi_1''+p\varphi_1'+q\varphi_1)v+v''\varphi_1+2v'\varphi_1'+pv'\varphi_1=0$

$\Rightarrow \underbrace{v''\varphi_1+v'(2\varphi_1'+p\varphi_1)=0}_{[※]}$ The equation above can be solved for v' using the method of integrating factor and is a first order O.D.E.

Let P be an anti-derivative of p. If $\varphi_1(t)\neq 0$ for all $t\in I$, then [※] implies that $(\varphi_1^2(t)e^{P(t)}v'(t))'=0 \Rightarrow \varphi_1^2(t)e^{P(t)}v'(t)=C_1 \Rightarrow$ $\varphi_1^2(t)v'(t)=C_1e^{-P(t)}$ for all $t\in I$. Now, $W(\varphi_1,\varphi_2)(t)=\det\begin{pmatrix}\varphi_1(t) & \varphi_2(t)\\ \varphi_1'(t) & \varphi_2'(t)\end{pmatrix}$

$=\det\begin{pmatrix}\varphi_1(t) & v(t)\varphi_1(t)\\ \varphi_1'(t) & v'(t)\varphi_1(t)+v(t)\varphi_1'(t)\end{pmatrix}=v'(t)\varphi_1^2(t)=C_1e^{-P(t)}\neq 0.$

Hence, $\{\varphi_1, v\varphi_1\}$ is a fundamental set of $y''+p(t)y'+q(t)y=0$.

Example 3.13: Given that $y=\varphi_1(t)=\frac{1}{t}$ is a solution of $2t^2y''+3ty'-y$ for $t>0$. Find a fundamental set of the equation. Let $\varphi_2(t)=\frac{v}{t}$ be another solution. Rewrite the equation: $y''+\frac{3}{2t}y'-\frac{1}{2t^2}y=0$. Then implies that $\frac{v''(t)}{t} + v'\left(-\frac{2}{t^2} + \frac{3}{2t^2}\right) = 0 \Rightarrow \frac{v''(t)}{t} - \frac{v'(t)}{2t^2} = 0$

$\Rightarrow v''(t) - \frac{v'(t)}{2t} = 0$. $-\frac{1}{2}\log t$ is an anti-derivative of $-\frac{1}{2t}$. We have

$e^{-\frac{1}{2}\log t} v''(t) - e^{-\frac{1}{2}\log t}\frac{v'(t)}{2t} = 0 \Rightarrow \frac{d}{dt}\left(e^{-\frac{1}{2}\log t} v'(t)\right) = 0 \Rightarrow e^{-\frac{1}{2}\log t} v'(t) = c$

$\Rightarrow v'(t) = c_1 e^{\frac{1}{2}\log t} = c_1\sqrt{t} \Rightarrow v(t) = c_1 t^{\frac{3}{2}} + c_2 \Rightarrow \varphi_2 = \sqrt{t}$ is a solution

of $2t^2y'' + 3ty' - y = 0$. $W(\varphi_1, \varphi_2)(t) = \det\begin{pmatrix} \frac{1}{t} & \sqrt{t} \\ -\frac{1}{t^2} & \frac{1}{2\sqrt{t}} \end{pmatrix} = \frac{3}{2}t^{-\frac{3}{2}} \neq 0$ for

all $t > 0$. Hence $\{\frac{1}{t}, \sqrt{t}\}$ is a fundamental set of $2ty'' + 3ty' - y = 0$. ▨

§3.5: Nonhomogeneous equations. In this section, we focus on solving the O.D.E: $y'' + p(t)y' + q(t)y = g(t)$. $\cdots$ (a).

Def 3.14: A particular solution to (a) is a solution of (a).

The space of complementary solutions to (a) is the collection of solutions to the corresponding homogeneous equation $y''(t) + p(t)y'(t) + q(t)y(t) = 0$. $\cdots$ (b) ▨

Let $y = Y(t)$ be a particular solution to (a). If $y = \varphi(t)$ is another solution to (a), then $y = \varphi(t) - Y(t)$ is a function in the space of complementary solutions to (a). By Thm 3.10, there exis $\varphi_1(t), \varphi_2(t)$ such that $\varphi(t) - \Upsilon(t) = c_1\varphi_1(t) + c_2\varphi_2(t)$.

Thm 3.15: The general solution of the nonhomogeneous equation (a) can be written in the form $\varphi(t) = C_1\varphi_1(t) + C_2\varphi_2(t) + \Upsilon(t)$, where $\{\varphi_1, \varphi_2\}$ is a fundamental set of (b), C_1, C_2 are arbitrary constants and $\Upsilon(t)$ is a particular solution of (a).

$\square$

General strategy of solving nonhomogeneous equation (a):

① Find fundamental set of (b).

② Find a particular solution $\Upsilon(t)$ of (a).

③ Apply Thm 3.15.

$\square$

I. ※※

§3.51: Method of variation parameters. This method can be used to solve a nonhomogeneous O.D.E when one solution to the corresponding homogeneous equation is known. Consider $y'' + p(t)y' + q(t)y = g(t)$ ... (a)

Suppose that we are given one solution $y = \varphi_1(t)$ to the corresponding homogeneous equation $y'' + p(t)y' + q(t)y = 0$ ... (b). Using the procedure in Section 3.4, we can find another solution $y = \varphi_2(t)$ to (b) so that
(the variantion of constant)
$\{\varphi_1, \varphi_2\}$ forms a fundamental set of (b). Our next goal is to obtain a particular solution to (a). Suppose $Y(t) = u(t)\varphi_1(t)$ is a particular solution of (a). Then similar computations as in Section 3.4 show that $u''\varphi_1 + u'(2\varphi_1' + p\varphi_1) = g$. Let P be an anti-derivative of p.

$\Rightarrow \varphi_1^2 e^P u'' + u'(2\varphi_1'\varphi_1 e^P + e^P p\varphi_1^2) = \varphi_1 g e^P \Rightarrow (\varphi_1^2 e^P u')' = \varphi_1 g e^P$.

$\Rightarrow \varphi_1^2(t) e^{P(t)} u'(t) = \int_{t_0}^t \varphi_1(s) g(s) e^{P(s)} ds \Rightarrow u'(t) = \dfrac{\int_{t_0}^t \varphi_1(s) g(s) e^{P(s)} ds}{\varphi_1^2(t) e^{P(t)}}$

$\Rightarrow u(t) = \int_{t_0}^t \left(\dfrac{\int_{t_0}^s \varphi_1(x) g(x) e^{P(x)} dx}{\varphi_1^2(s) e^{P(s)}} \right) ds \Rightarrow$ A particular solution is of the form $Y(t) = \varphi_1(t) \int_{t_0}^t \left(\dfrac{\int_{s_0}^s \varphi_1(x) g(x) e^{P(x)} dx}{\varphi_1^2(s) e^{P(s)}} \right) ds$. ... (C)

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Lecture VI.

II. ※※

Example 3.16: As Example 3.13, let $y = \varphi_1(t) = \frac{1}{t}$ be a given solution to $2t^2y'' + 3ty' - y = 0$ for $t > 0$. Suppose that we are looking for solutions to $2t^2y'' + 3ty' - y = 2t^2$ for $t > 0$. $\Rightarrow y'' + \frac{3}{2t}y' - \frac{y}{2t^2} = 1$ for $t > 0$. $\frac{3}{2}\log t$ is an anti-derivative of $\frac{3}{2t}$. By using (c), we know that a particular solution is given by $Y(t) = \frac{1}{t}\int_0^t \left(\frac{\int_0^s \frac{1}{x} e^{\frac{3}{2}\log x} dx}{\frac{1}{s^2} e^{\frac{3}{2}\log s}} \right) ds = \frac{1}{t}\int_0^t \left(\frac{\int_0^s x^{\frac{1}{2}} dx}{\frac{1}{s^{\frac{1}{2}}}} \right) ds = \frac{1}{t}\int_0^t s^{\frac{1}{2}} \int_0^s x^{\frac{1}{2}} dx = \frac{1}{t}\int_0^t s^{\frac{1}{2}} \frac{2}{3} s^{\frac{3}{2}} ds = \frac{2}{9}t^2$. Recall that $\{\frac{1}{t}, \sqrt{t}\}$ is a fundamental set of the O.D.E. $2t^2y'' + 3ty' - y = 0$. Hence all the solutions of $2t^2y'' + 3ty' - y = 2t^2$ for $t > 0$ can be written as

$$Y(t) = \frac{C_1}{t} + C_2\sqrt{t} + \frac{2}{9}t^2.$$

$\square$

Let $\{\varphi_1, \varphi_2\}$ be a fundamental set of (b). We can also look for a particular solution of (a) of the form $Y(t) = C_1(t)\varphi_1(t) + C_2(t)\varphi_2$. Plugging such Y in (a), we find that $C_1''\varphi_1 + C_1'(2\varphi_1' + p\varphi_1) + C_2''\varphi_2 + C_2'(2\varphi_2' + p\varphi_2) = g$. (d) Assume $C_1'\varphi_1 + C_2'\varphi_2 = 0$ (*). $\Rightarrow C_1'\varphi_1 + C_2'\varphi_2 = 0 \Rightarrow C_1''\varphi_1 + C_2''\varphi_2 = -C_1'\varphi_1' - C_2'\varphi_2'$, thus (d) reduces to $C_1'\varphi_1' + C_2'\varphi_2' = g$.

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$$\Rightarrow \begin{cases} C_1'\varphi_1 + C_2'\varphi_2 = 0 \\ C_1'\varphi_1' + C_2'\varphi_2' = g. \end{cases} \Rightarrow C_1'(t) = \frac{-g(t)\varphi_2(t)}{W(\varphi_1,\varphi_2)(t)}, \quad C_2'(t) = \frac{g(t)\varphi_1(t)}{W(\varphi_1,\varphi_2)(t)}$$

$$C_1'(t) = \frac{\begin{vmatrix} 0 & \varphi_2 \\ g & \varphi_2' \end{vmatrix}}{\begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix}}, \quad C_2'(t) = \frac{\begin{vmatrix} \varphi_1 & 0 \\ \varphi_1' & g \end{vmatrix}}{\begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix}}$$

Thm 3.17: If the functions p, q, g are continuous in an open interval I and $\{\varphi_1, \varphi_2\}$ be a fundamental set of (b). Then a particular solution to (a) is $Y(t) = -\varphi_1(t)\int_{t_0}^{t} \frac{g(s)\varphi_2(s)}{W(\varphi_1,\varphi_2)(s)}\,ds + \varphi_2(t)\int_{t_0}^{t} \frac{g(s)\varphi_1(s)}{W(\varphi_1,\varphi_2)(s)}\,ds$, ... (e) where $t_0 \in I$ can be arbitrarily chosen and the general solution to (a) is $y(t) = C_1\varphi_1(t) + C_2\varphi_2(t) + Y(t)$. ▨

Example 3.18: $\varphi_1(t) = \frac{1}{t}$, $\varphi_2(t) = \sqrt{t}$ is a fundamental set of $2t^2y'' + 3ty' - y = 0$, $t > 0$. To solve $2t^2y'' + 3ty' - y = 2t^2$ for $t > 0$, we need to find a particular solution. Rewrite $y'' + \frac{3}{2t}y' - \frac{y}{2t^2} = 1$.

$g = 1$, $W(\varphi_1, \varphi_2) = \frac{3}{2}t^{-\frac{3}{2}}$. By (e), $Y(t) = -\frac{1}{t}\int_0^t \frac{\sqrt{s}}{\frac{3}{2}s^{-\frac{3}{2}}}\,ds + \sqrt{t}\int_0^t \frac{\frac{1}{s}}{\frac{3}{2}s^{-\frac{3}{2}}}\,d$

$= \frac{2}{9}t^2$ is a particular solution of $2t^2y'' + 3ty' - y = 2t^2$, for $t > 0$. ▨

§4.1: General Theory of n-th order linear equation. An n-th order linear ordinary equations is an equation of the form: $P_n(t)y^{(n)}(t) + P_{n-1}(t)y^{(n-1)}(t) + \cdots + P_1(t)y^{(1)}(t) + P_0(t)y(t) = G(t)$, where P_n is never zero.

IV. ※※

Lecture VII.

Divide both sides by $P_n(t)$, we obtain $L[y] := y^{n}(t) + P_{n-1}(t)y^{n-1}(t) + \cdots + P_0(t)y(t) = g(t)$. ⋯ [a] Suppose $P_j \equiv 0$, $j = 0, 1, \cdots, n-1$. Then to determine y, it requires n times integration and each integration produce an arbitrary constant. Therefore, we expect that to determine the solut y to [a] uniquely, it requires n initial conditions: $y(t_0) = y_0$, $y'(t_0) = \cdots$, $y^{(n-1)}(t_0) = y_{n-1}$, [b] where t_0 is some point in an open interval I, and $y_0, y_1, \cdots, y_{n-1}$ are some given constants. [a] is called homogeneous $g \equiv 0$.

Thm 4.1: If the functions $P_0, P_1, \cdots, P_{n-1}$ and g are continuous on an open interval I, then there exists exactly one solution $y = \varphi(t)$ of [with initial condition [b], where t_0 is any point of I. This solut exists throughout the interval I.

▯

Def 4.2: Let $\{\varphi_1, \cdots, \varphi_n\}$ be a collection of n differentiable function defined on an open interval I. The Wronskian of $\varphi_1, \varphi_2, \cdots, \varphi_n$ at

$t_0 \in I$, denoted by $W(\varphi_1, \cdots, \varphi_n)(t_0)$, is the number $W(\varphi_1, \cdots, \varphi_n)(t_0) =$

$\det \begin{pmatrix} \varphi_1(t_0), & \cdots, & \varphi_n(t_0) \\ \varphi_1'(t_0) & & \varphi_n'(t_0) \\ \varphi_1^{(n-1)}(t_0) & & \varphi_n^{(n-1)}(t_0) \end{pmatrix}$. ///

Thm 4.3 : Let $y = \varphi_1(t), \cdots, y = \varphi_n(t)$ be solutions to the homogeneous equation $L[y] := y^{(n)}(t) + P_{n-1}(t) y^{(n-1)}(t) + \cdots + P_1(t) y^{(1)}(t) + P_0(t) y(t) = 0$. $[C]$

Then, $\frac{\partial}{\partial t} W(\varphi_1, \cdots, \varphi_n)(t) + P_{n-1}(t) W(\varphi_1, \cdots, \varphi_n)(t) = 0$. ///

pf: Let $n=2$. Then $\frac{\partial}{\partial t} \det \begin{pmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{pmatrix} = \frac{\partial}{\partial t} (\varphi_1 \varphi_2' - \varphi_1' \varphi_2) = \varphi_1 \varphi_2'' - \varphi_1'' \varphi_2$

$= \det \begin{pmatrix} \varphi_1 & \varphi_2 \\ \varphi_1'' & \varphi_2'' \end{pmatrix}$. Claim that $\frac{\partial}{\partial t} W(\varphi_1, \cdots, \varphi_n) = \det \begin{pmatrix} \varphi_1, & \cdots, & \varphi_n \\ \varphi_1' & & \varphi_n' \\ \varphi_1^{(n-2)} & & \varphi_n^{(n-2)} \\ \varphi_1^{(n)} & & \varphi_n^{(n)} \end{pmatrix}$.

By the discussion above, we see that the claim holds for $n=2$.

Assume that the claim holds for $n \leq k$. Let $n = k+1$. $W(\varphi_1, \cdots, \varphi_{k+1})$

$= \varphi_1 \det \begin{pmatrix} \varphi_2', & \varphi_{k+1}' \\ \varphi_2^{(k)} & \varphi_{k+1}^{(k)} \end{pmatrix} - \varphi_2 \det \begin{pmatrix} \varphi_1' & \varphi_3', & \varphi_{k+1}' \\ \varphi_1^{(k)} & \varphi_3^{(k)} & \varphi_{k+1}^{(k)} \end{pmatrix} + \cdots + (-1)^k \varphi_{k+1} \det \begin{pmatrix} \varphi_1' & \varphi_k' \\ \varphi_1^{(k)} & \varphi_k^{(k)} \end{pmatrix}$.

By induction, $\frac{\partial}{\partial t} W(\varphi_1, \varphi_{k+1}) = \varphi_1' \det \begin{pmatrix} \varphi_2', & \cdots, & \varphi_{k+1}' \\ \varphi_2^{(k)}, & & \varphi_{k+1}^{(k)} \end{pmatrix} - \varphi_2' \det \begin{pmatrix} \varphi_1', & \varphi_3', & \cdots, & \varphi_{k+1}' \\ \varphi_1^{(k)} & \varphi_3^{(k)}, & \cdots & \varphi_{k+1}^{(k)} \end{pmatrix}$

$+ \cdots + (-1)^k \varphi_{k+1}' \det \begin{pmatrix} \varphi_1' & \cdots, & \varphi_k' \\ \varphi_1^{(k)} & \cdots & \varphi_k^{(k)} \end{pmatrix} + \varphi_1 \det \begin{pmatrix} \varphi_2' & \cdots & \varphi_{k+1}' \\ \varphi_2^{(k-1)} & & \varphi_{k+1}^{(k-1)} \\ \varphi_2^{(k+1)}, & & \varphi_{k+1}^{(k+1)} \end{pmatrix} \cdots$

$= \underbrace{\det \begin{pmatrix} \varphi_1' & \cdots & \varphi_{k+1}' \\ \varphi_1' & & \varphi_{k+1}' \\ \varphi_1^{(k)} & & \varphi_{k+1}^{(k)} \end{pmatrix}}_{\substack{\parallel \\ 0}} + \det \begin{pmatrix} \varphi_1 & , & \varphi_{k+1} \\ \varphi_1' & & \varphi_{k+1}' \\ \varphi_1^{(k-1)} & & \varphi_{k+1}^{(k-1)} \\ \varphi_1^{(k+1)} & & \varphi_{k+1}^{(k+1)} \end{pmatrix}$. The claim follows.

VI. ※※

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$$\Rightarrow \frac{\partial}{\partial t} W(\varphi_1, \cdots, \varphi_n) = \det\begin{pmatrix} \varphi_1 & \cdots, & \varphi_n \\ \vdots & & \vdots \\ \varphi_1^{(n-2)} & & \varphi_n^{(n-2)} \\ \varphi_1^{(n)}, & & \varphi_n^{(n)} \end{pmatrix} = \det\begin{pmatrix} \varphi_1 & & \varphi_n \\ \vdots & & \vdots \\ \varphi_1^{(n-2)} & \cdots, & \varphi_n^{(n-2)} \\ -p_{n-1}\varphi_1^{(n-1)} \cdots p_0\varphi_1, & \cdots & -p_{n-1}\varphi_n^{(n-1)} \cdots p_0\varphi_n \end{pmatrix} =$$

$$-p_{n-1} W(\varphi_1, \cdots, \varphi_n).$$